The Riemann Hypothesis:

Probability, Physics, and Primes

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Abstract

This paper is an introduction to the Riemann Hypothesis and the related Riemann Zeta function. We discuss what the Hypothesis is and why it has remained a pertinent mathematical question for 155 years. In addition, we cover the Riemann Hypothesis's history, its implications in various fields of science, and popular methods for approaching it. This paper also includes tutorials to related concepts in mathematics and physics as well as proofs of selected mathematical results.

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Introduction

The Riemann Hypothesis is one of the most famous math problems of all time. It was first suggested in 1859 and remains unproven to this day, despite being one of the most fascinating problems in the world for mathematicians. In 1900, it was chosen by German mathematician David Hilbert to be on his list of 23 key problems from various fields of mathematics, and in 2000, it was selected to be one of the seven Millennium Problems—the first person to prove or disprove the Riemann Hypothesis will receive one million dollars.

Essentially, the Riemann Hypothesis guarantees the randomness of the prime numbers—if it is false, then there will be some pattern to the primes' distribution among the positive integers (du Sautoy, 167). In addition, if the Riemann Hypothesis is true, many mathematical formulas will be true, including one that tells us how the number of prime numbers less than a certain quantity behaves (Sabbagh, 29).

This means that the Riemann Hypothesis is crucial to number theory (the study of integers) and to math in general. But this does not mean that it should only be treated as a "pure" math problem. The Riemann Hypothesis is also related to applied math and science—particularly fields such as statistics and physics. Because of this, ideas stemming from the fields of probability theory or the study of subatomic particles could very well be the key to solving the Riemann Hypothesis—and, by extension, to the multitudes of other math problems that are similar to the Riemann Hypothesis. Thus, although mathematicians pursue a proof of Riemann Hypothesis, they do not merely seek to say "yes, it is true." Rather, they search for new insights and reasoning with which to approach this problem—and hopefully, future problems.

About Riemann

The Riemann Hypothesis takes its name from Georg Friedrich Bernhard Riemann, who was born September 17, 1826, in Hanover, Germany. From an early age, Riemann was interested in mathematics: at six, he invented problems for his teachers, while at ten he learned advanced arithmetic and geometry from university professors—and often had better solutions than they did (Hunsicker)! When he was fourteen, Riemann began attending gymnasiums, or college preparatory schools, where his teachers noticed and nurtured his talent. One teacher is said to have lent him a dense mathematical book (of 859 pages) on number theory—Riemann mastered the book in six days (du Sautoy, 62).

Later in life, Riemann studied and worked at the University of Göttingen, where he continued to demonstrate impressive mathematical skill. For example, when at the age of 28 he applied for the position of unpaid lecturer, he had to give a trial lecture to the faculty of the school—which included Carl Friedrich Gauss. Even in the 1800s, Gauss was recognized as one of the most famous mathematicians and physicists of all time, an expert in fields such as geometry, number theory, astronomy, and electromagnetism. For example, he proved the fundamental theorem of algebra, helped to invent the electromagnetic telegraph, and calculated the position of the asteroid Ceres when astronomers lost track of it in 1801 (Simmons, 178-181). Moreover, at the time of Riemann's lecture Gauss had been thinking about Riemann's topic—the foundations of geometry—for the past sixty years. Yet even Gauss was surprised and impressed by Riemann's insight on the topic. In fact, Riemann's ideas were eventually found to be crucial not only to pure mathematics, but also to Einstein's General Theory of Relativity (Simmons, 203).

In number theory, Riemann wrote only one paper, and a very brief one at that—it is only nine pages long. Yet in “On the Number of Prime Numbers less than a Given Quantity,” which he published in 1859, Riemann made several highly profound remarks about the Prime Number Theorem. The first version of the Prime Number Theorem was proposed by Gauss when he was only fifteen years old; a second variation was suggested by the French mathematician Adrien-Marie Legendre in 1808, and a third form was discovered by Gauss around 1849 but not published until after his death. This theorem says that for larger and larger quantities, the number of prime numbers less than or equal to that quantity—the prime counting function—becomes closer and closer to a certain expression. In his 1859 paper, Riemann stated an expression slightly different from any of Legendre’s or Gauss’s, believing that it was more accurate. (It turns out that this is true for many numbers, but sometimes Gauss’s approximation for the prime counting function is still the more exact (du Sautoy, 129).) However, Riemann did not realize that if a certain hypothesis—the now-called Riemann hypothesis—in his paper is true, we will know not only the expression that the prime counting function approaches, but also the amount with which it can differ from that expression. (A formula for this amount was discovered in 1901 by the Swedish mathematician Helge Von Koch, who we will mention again later.)

Although Riemann’s work was of extremely high quality, we have very little of it. Not only did Riemann die when he was thirty-nine, he only published a few papers during his lifetime—he was a perfectionist, refusing to share ideas until he was sure they were flawless (Hunsicker). It is true that he had many more ideas than those he revealed to the public, but after he died, many of his unpublished notes were deemed junk and destroyed by his housekeeper. Moreover, many of Riemann’s rough drafts and other papers contained both mathematical ideas and personal notes, which made his family reluctant to publish those of his writings that remained. Over the years some of Riemann's papers made their way to libraries, but many others have been irrevocably lost. (du Sautoy, 101, 153-154). This means that the mathematical community will never fully know what Riemann was capable of. Yet it is true that what work he did leave behind was highly important. He interlinked analysis (the field of mathematics involving limits) with geometry. He created the Riemann integral, a fundamental part of calculus. His ideas on the foundations of geometry proved to represent space as it is described by general relativity. And he invented the Riemann zeta function, which is related to statistics, physics, and prime numbers. The Riemann Hypothesis, one of the most important problems of all time, revolves around Riemann's zeta function:

The Riemann Hypothesis:

All non-trivial zeros of the zeta function have real part one-half.

A Brief Tutorial

This section will provide an introduction to some mathematical concepts and notations that are important components of the Riemann zeta function and the Riemann Hypothesis: functions, the different types of numbers, the concept of infinite sums and products, exponentiation, and prime numbers.

Functions

A ***function*** is a mathematical procedure that takes in a certain variable (number) and returns a unique value. For example, we can define a function f to return the square of its input: $f(x) = x^2$. (Note that $f(x)$, read as "f of x," is the *value* that the function f returns when we input the variable x.) Sometimes we can leave out the parentheses around the ***argument***, or input variable. For example, given that ln is the symbol for the natural logarithm function and x is the argument, we write the natural logarithm of x as $\ln x$.

The Types of Numbers

Numbers can be divided into multiple classes and subclasses:

- Integers: numbers such as $-1000,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 1000$
- Rational numbers: integers together with fractions (both positive and negative)
- Irrational numbers: numbers which are not integers and cannot be written as fractions. Examples include $\sqrt{2}, \pi,$ and e. (We can approximate these three numbers by $1.41421,$ $3.14159,$ and 2.71828 , respectively, but they lack *exact* decimal or fractional representations.)
- Real numbers: rational and irrational numbers.

At this point it may seem as if we have covered all numbers which exist. But there are other classes of numbers:

- Imaginary numbers: these numbers can be written as the square roots of negative numbers—yes, square roots of negative numbers. If we limit ourselves to thinking about real numbers, this seems impossible, since we know that a number multiplied by itself is either zero or positive (a positive times itself is positive, and a negative times itself is also positive). But if we expand our thinking somewhat, we can envision a number that is equal to the square root of negative one and assign it a symbol: i.

$$i = \sqrt{-1}$$

Then, other imaginary numbers can be written as i multiplied by some real number. For example, we can say that $\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3i$, and that $\sqrt{-\frac{1}{4}} = \sqrt{\frac{1}{2} \times \frac{1}{2}} \times \sqrt{-1} = \frac{1}{2}i$.

- Complex numbers: the largest class into which numbers can be categorized. Complex numbers include *all* numbers, including those that are purely real and those that are purely imaginary. The general form of a complex number is $z = a + bi$, where a is the real part of z while b (*not* bi) is the imaginary part. Another way to express a and b is $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. Note that the numbers a and b themselves are real. (So, if $z = (3 + i) + 2i = 3 + 3i,$ we write $a = 3,\ b = 3,$ not $a = 3 + i,\ b = 2$.)

  Under this notation, the real number 4 can also be considered complex since we can write it as $4 + 0i$. Similarly, an imaginary number is also considered to be complex (for example, $9i = 0 + 9i$).

The Concept of Infinity

Our discussion of the Riemann zeta function will involve both infinite sums and products. An example of such a sum would be

$$\sum_{n} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

that is, "the sum over all n of 1 over n." "All n," though, is usually interpreted to mean "all positive whole numbers": 1, 2, 3, 4, and so on.

In the above expression, Σ is the capital Greek letter sigma. We use it to denote a sum (sigma is the equivalent of our letter s.) Similarly, we use a capital pi (the Greek equivalent of p) to denote a product, as in

$$\prod_n n = 1 \times 2 \times 3 \times 4 \times \ldots$$

In both of the above expressions, we have dealt with infinite sums or products—n takes on an infinite number of values. Now, infinity can be a confusing concept. A classic example is the answer to the question "What is two times infinity?" The answer, of course, is infinity. But we know that no number other than zero can equal two times itself. Thus infinity is not a specific number like 10^{100} (1 with a hundred zeros). Because of this, mathematicians do not say that something "equals" or "is" infinity, as these terms imply that infinity is a specific number. Instead, they say that a very large number "approaches infinity." (For practical purposes, though, mathematicians often write that an expression $= \infty$.)

It is fairly apparent that an infinite sum can approach infinity. For example,

$$\sum_n n = 1 + 2 + 3 + 4 + 5 + \cdots$$

increases without bound; it approaches infinity.

But now evaluate

$$\sum_n 0$$

Even though we have added up an infinite number of zeros, the sum is still zero.

An infinite sum can also be exactly equal to a number other than infinity, negative infinity, or zero. As an example,

$$\sum_n \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

is exactly equal to 1. This is possible because although we add up an infinite number of positive quantities, these quantities eventually become infinitesimally small, and the balance between the two extremes of infinity is such that the sum is a finite, positive number. Infinite products can behave in a similar way; just because we multiply an infinite number of quantities does not mean that the final answer is zero or infinity.

If an infinite sum or product is exactly equal to some finite number (including, potentially, 0), then we say that it ***converges***.

Exponentiation

Exponentiation is the process of multiplying together a certain number of identical numbers. This is denoted by a^x, or "a raised to the power of x." In this expression, a is the ***base*** (the quantity being multiplied) and x is the ***exponent*** (the number of a's that are multiplied together). For example, $2^3 = 2 \times 2 \times 2 = 8$.

It looks as if exponentiation defined in this way is limited. After all, how do you multiply 2 by itself $\frac{1}{2}$ or -3 times?

So mathematicians have extended this definition to apply to exponentiation with *any* base and *any* exponent. Exponentiation revolves around two rules, which hold for any numbers a, x, and y (note that these numbers do not even have to be real):

Rule 1: $a^x \times a^y = a^{x+y}$

Rule 2: $(a^x)^y = a^{xy}$

By using these rules, we can deal with any real exponent. For example:

- $a^0 = 1$ for all a except for $a = 0$. This is because according to Rule 1, $a^1 \times a^0 = a^{1+0} = a^1$, and thus (since a^1 is simply a)

  $$a^0 = \frac{a}{a} = 1$$

  The exception occurs when $a = 0$ since then

$$0^0 = \frac{0}{0}$$

which is undefined due to the division by 0.

- $a^{1/n}$ is the nth root of a, if n is a positive integer. This is because according to Rule 2, $(a^{1/n})^n = a^1 = a$, so $a^{1/n}$ is the number which, when multiplied by itself n times, results in a.
- $a^{-n} = \frac{1}{a^n}$ for all $a \neq 0$ and any number n, since by Rule 1, we must have $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$. (Again, the exception occurs when $a = 0$ because we cannot divide 1 by $0^n = 0$.)

So we can evaluate, for example, $16^{-0.75}$. By our logic above,

$$16^{-0.75} = 16^{-3/4} = \frac{1}{16^{3/4}}$$

According to Rule 2,

$$16^{3/4} = \left(16^{1/4}\right)^3 = 2^3 = 8$$

since 2 is the fourth root of 16. Then

$$16^{-0.75} = \frac{1}{16^{3/4}} = \frac{1}{8}$$

Exponentiation is also defined for nonreal exponents: in general, if we have the complex number $z = x + iy$ and a real number a, then

$$a^z = Y \times a^x$$

where Y is a complex number that depends on the value of y.

We can even define exponentiation for nonreal bases—in fact, there is a general definition for exponentiation with a complex base *and* complex exponent. However, this is beyond the scope of this paper, and is not required to understand the zeta function. For now, suffice to say that it is possible to raise a positive integer n to the power of a complex number s. This is a key part of the definition of the Riemann zeta function.

Prime Numbers

A ***prime number*** (sometimes called a "prime integer" or simply a "prime" (Weisstein, "Prime Number")) is a positive integer that can be divided by exactly two positive integers: 1 and itself. Positive, non-prime integers are called ***composite***, with the exception of 1—it is neither prime nor composite[1].

The first few primes are $2, 3, 5, 7, 11, 13, 17, 19,$ 23, and 29. One interesting fact about primes is that there are infinitely many of them, as was shown by the Ancient Greek mathematician Euclid around 300 B.C.E. We include Euclid's proof of this result as Proof 1 in Appendix A.

The Riemann Zeta Function

Riemann's 1859 paper revolved around ζ, the ***Riemann zeta function***, or, as it is sometimes called, *the* zeta function. (The Greek letter "zeta" can be pronounced either "ZAY-ta" or "ZEE-ta".) Following the notation that Riemann originally used, we typically use the symbol s to indicate the zeta function's argument. For certain values of s, we define zeta of s as

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

that is, "the sum over all n of one over n to the s." Other, more general definitions of the zeta function[2] exist, but none of them are quite as elegant and simple as this one.

Riemann was not the first important mathematician to do work on the zeta function—as early as 1737, the Swiss mathematician Leonhard Euler was studying it (Simmons, 294). (Euler, pronounced "OI-ler," lived from 1707 to 1783. He invented much of our mathematical notation and was so important to mathematics that some say that modern-day calculus textbooks are all copies of Euler (Simmons, 162).) In 1737, Euler discovered that when s is a real number greater than 1,

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

where the product on the right is taken over all prime numbers (recall that there are infinitely many of them). A derivation of Euler's result is provided as Proof 2 in Appendix A.

Riemann continued Euler's work—he extended the definition of the zeta function from situations in which s must be real to more general cases in which s is a complex number, and it is for this accomplishment that *the* zeta function is named after Riemann. Moreover, Riemann noticed that the above relationship between $\zeta(s)$ and prime numbers can be generalized to situations where s is not real; however, the real part of s must be greater than 1, or the sum on the left side will not converge. (For a derivation of this general relationship, see Proof 3 in Appendix A.) Despite this restriction on s, this relationship was still a very important result: it relates $\zeta(s)$ to the prime numbers. In fact, it was the first inkling of the intricate relationship between the zeta function's zeros and the prime numbers.

What is the Riemann Hypothesis?

The Riemann Hypothesis states that all non-trivial zeros of the zeta function have real part one-half. We have already seen a definition of the zeta function. What, then, are the ***non-trivial*** zeros of the zeta function? The ***zeros*** or ***roots*** of $\zeta(s)$ are the values of s such that $\zeta(s) = 0$. (Either term is equally valid; for the sake of consistency, we will exclusively use the term "zero.") A ***trivial*** zero is one that we can see with comparatively little work. For example, if we wanted to find the zeros of the function $(s-1)\zeta(s)$, an obvious (trivial) zero would be $s = 1$, since this makes $s - 1 = 0$ and so $(s-1)\zeta(s) = 0 \times \zeta(s) = 0$. (Naturally, the term "trivial" is subjective—trivial zeros are *comparatively* obvious to find.)

Returning to the Riemann Hypothesis, we see that it makes a statement about the zeta function's non-trivial zeros. This raises the question—what are the trivial zeros? It turns out that the trivial zeros of the zeta function are just all the negative, even integers ($-2,\ -4,\ -6,\ -8,$ ...). This seems like a contradiction. After all,

$$\zeta(s) = \sum_n \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

If we substitute $s = -2$, we have

$$\zeta(-2) = \sum_n \frac{1}{n^{-2}} = \sum_n \frac{1}{\frac{1}{n^2}}$$

$$= \sum_n n^2 = 1^2 + 2^2 + 3^2 + \cdots$$

This is an infinite sum of positive numbers, which is certainly not zero. Unfortunately, this expression for $\zeta(s)$ is not valid when $s = -2$.

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

only holds when the real part of s is greater than 1—for all other numbers, we must use a more explicit form of $\zeta(s)$, which will be valid in general (in fact, it is true for all s except for $s = 1$).

One formula for $\zeta(s)$, first expressed by Riemann in his 1859 paper, involves calculus, so we will not analyze it here. However, it is also possible to solve for $\zeta(s)$ in terms of $\zeta(1-s)$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-\mathrm{s})\zeta(1-s)$$

(Note that Γ, the Greek letter gamma, represents a function.) This is called the ***functional equation*** of the zeta function.

This equation holds for all s (except for $s = 1$; $\zeta(1)$ can never be defined). Using this general equation, we can show, as we do in the Appendix B, that when s is a negative multiple of 2, $\zeta(s)$ is 0. Thus the trivial zeros of the zeta function are universally defined to be all the negative multiples of 2.

We can show the difference between the trivial and non-trivial zeros of the zeta function by plotting their values in the ***complex plane***, which is a coordinate plane resembling our ordinary xy-plane. However, instead of the x-axis, we have the real axis; instead of the y-axis, we have the imaginary axis. The complex number $z = \sigma_0 + it_0$ (σ is the Greek lowercase letter sigma[3] and σ_0 is pronounced “sigma-nought”) is plotted as below in Figure 1:

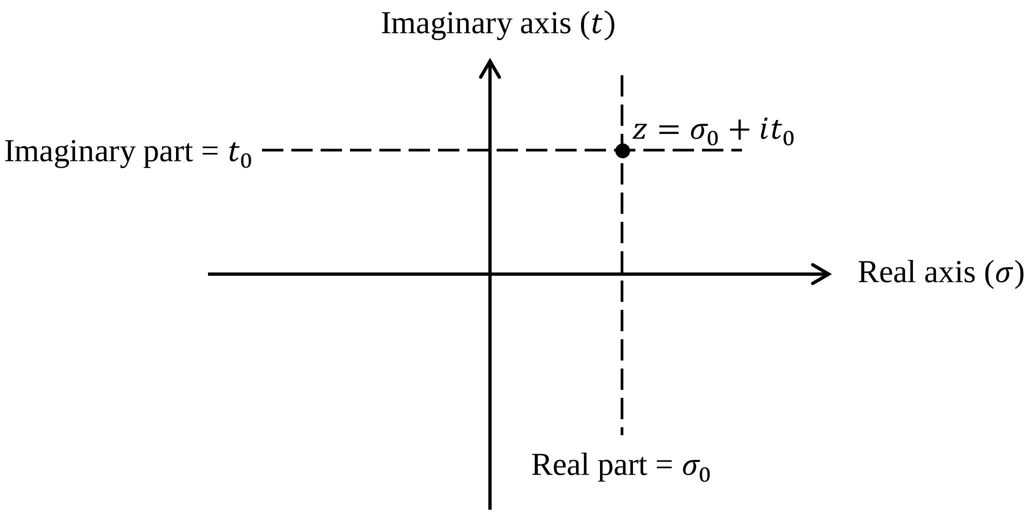
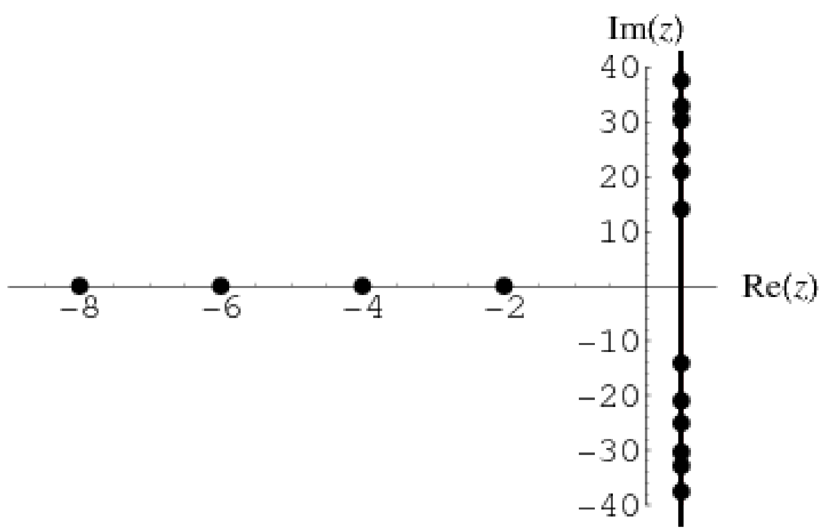


Figure 1: the number $z = \sigma_0 + it_0$ plotted in the complex plane.

When we graph the zeros of $\zeta(s)$ in the complex plane, the trivial zeros are all real, so they lie on the real axis (the horizontal line). The other points in Figure 2 are the non-trivial zeros of the zeta function; if the Riemann Hypothesis is true, then they should all have real part $\frac{1}{2}$. In other words, they should lie on the ***critical line***, the vertical line $\sigma = \frac{1}{2}$ (here depicted in bold).



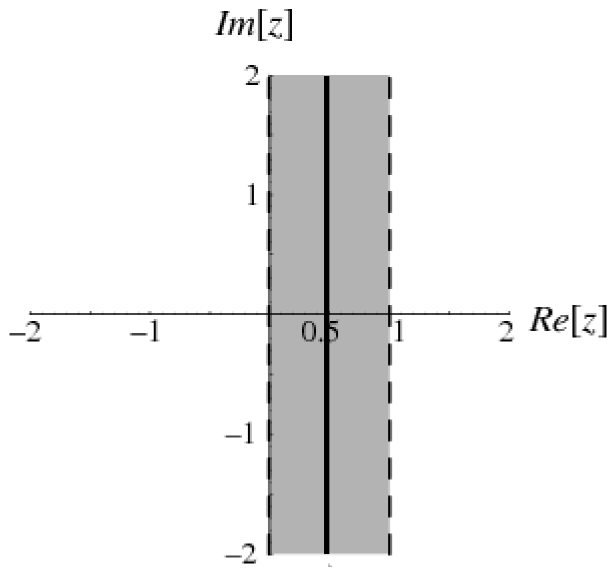
Figure[4] 2: The zeros of the zeta function in the complex plane.

Note that the Riemann Hypothesis only concerns itself with non-trivial zeros. As a result, we will from here on use the term “zeros of the zeta function” to refer to *non-trivial* zeros, unless otherwise stated.

Where to Look for Zeros?

Ever since the Riemann Hypothesis was first suggested, over 150 years ago, mathematicians have been trying to prove or disprove it. Those who wish to prove the Hypothesis must make sure that all possible zeros lie on the critical line; those who wish to disprove it must find at least one zero off the line. So, must we search the entire complex plane for zeros?

The answer turns out to be no—over the years various discoveries have established that zeros can only occur within a certain region: specifically, within the ***critical strip***. For example, it has been found that $\zeta(s)$ has no zeros with real part greater than 1 (see Proof 4 in Appendix A for a proof of this result). It is also known that the only zeros with real part less than zero are the trivial zeros—the negative even integers. (This result is proved in Proof 5 of Appendix A.) Moreover, no zeros lie on $\sigma = 0$ or $\sigma = 1$. Taken together, these statements mean that all non-trivial zeros of $\zeta(s)$ must lie in the critical strip $0 < \sigma < 1$. This considerably narrows the domain for possible zeros.

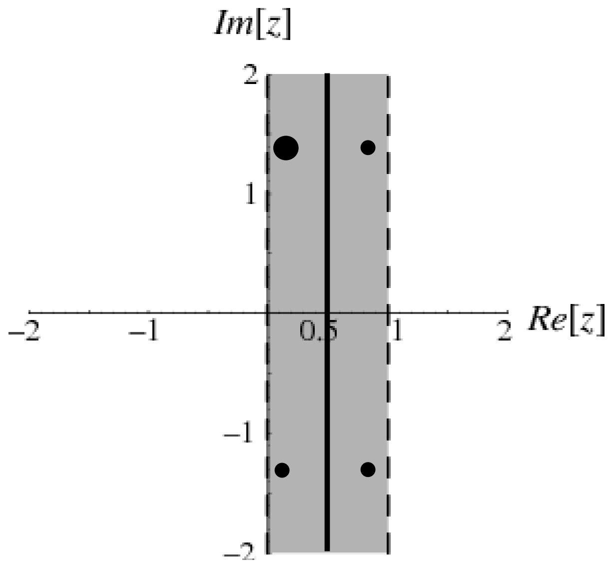


Figure[5] 3: The critical strip (shaded) and the critical line (in bold).

The dashed lines indicate that zeros cannot occur on those lines.

Also, all of the zeta function's zeros (both trivial and non-trivial) must be symmetrical about the real axis, and the non-trivial zeros of $\zeta(s)$ are symmetric about the critical line ($\sigma = \frac{1}{2}$). This information can be deduced from the zeta function's alternative forms and its functional equation (Goodman and Weisstein).

This means that if there is a single zero off of the line, there are actually three more "companion" zeros that do not lie on the line. (One exception occurs if this zero lies on the real axis: then there is only one "companion" zero. However, it has been confirmed that no zeros in the critical strip lie on the real axis, so we can ignore this special case.)



Figure[6] 4: a hypothetical zero that does not lie on the critical line (largest dot),

and its three "companion" zeros (smaller dots).

Because of this, when we search for zeros, we need only look within the critical strip and above the real axis—all zeros below the real axis have "companion" zeros above the real axis. In addition, if there is a zero to the left of the critical line, there will be a zero to the right, and vice versa.

Proofs, Disproofs, and Counterexamples

It may seem that disproving the Riemann Hypothesis is easier than proving it, since one would simply need to find a solution $s = \sigma + it$ to $\zeta(s) = 0$ that does not lie on the critical line—a counterexample. However, counterexamples are not necessarily disproofs.

It is true that the Hypothesis states that *all* non-trivial zeros of the zeta function should lie on the critical line—so if even one zero is not on the line, the Hypothesis as previously stated is false. Yet if the twenty-trillionth zero (and its three "companions") does not lie on the line, while all other zeros do, can't we say that the Hypothesis is *mostly* true, except in a few special cases?

The Clay Math Institute, which offers the million-dollar prize for a proof or disproof of the Hypothesis, makes this condition. If this situation was to happen, and the Riemann Hypothesis "[survived] after reformulation or elimination of some special case" ("Rules for the Millennium Prizes"), the Institute would not present the prize. It might give a smaller award, but not the million dollars—that prize is for a *rigorous* proof or disproof or for a counterexample that unquestionably proves the Hypothesis false ("Rules for the Millennium Prizes").

This distinction between a disproof and a counterexample is a fine but important one. Although proving—or disproving—the Riemann Hypothesis would be a massive accomplishment, mathematicians do not seek the result as much as they do the process. After all, as of right now we do not even know if the Hypothesis *can* be proved. Perhaps it is a statement similar to "x equals 1." We do not know what x is—it might be 1, or it might be $1{,}000$—and without further information we cannot come to a conclusion about the statement. In a similar way, maybe the Riemann Hypothesis actually *cannot* be shown to be true or false.

So mathematicians are really trying to find whether it is *possible* to prove the Riemann Hypothesis. Simply churning through all the zeros and testing if they lie on the critical line is useful for seeing if the Hypothesis holds true for the first few zeros, but in the long run, it is inelegant, and it is no proof. Similarly, finding a single zero that lies off the critical line is not necessarily a disproof. Of course, this does not mean a zero off the critical line will be ignored. Rather, if one is found, mathematicians will continue trying to discover if the Riemann Hypothesis is true *in general* or not—and, more importantly, *why* it is true in general or not.

The History of the Riemann Hypothesis

When Riemann first suggested his Hypothesis, he did not actually state that "all non-trivial zeros of the zeta function have real part one-half." Instead, after eliminating trivial zeros from consideration, he related $\zeta(s)$ to another function of the variable t, where $s = \frac{1}{2} + it$. Riemann then noted that $\zeta(s)$ would only have zeros when the second function was zero, and hypothesized that this could only happen when t was real…which would make s have real part one-half. Unfortunately, Riemann had no proof of this claim. "Certainly one would wish for a stricter proof here," he said. "I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation [finding a better version of the Prime Number Theorem[7]]" (Riemann, 4).

Over the years, no one has found a counterexample to Riemann's Hypothesis. Computers have found that the first ten trillion (1 with 13 0's following it) zeros of the zeta function *all* lie on the critical line (Goodman and Weisstein)—that is, for t ranging between 0 and about 2.4 trillion, no zeros lie off the critical line. However, the positions of the zeros after the first ten trillion or so are not known. Only if *all* zeros of the zeta function lie on the critical line is the Riemann Hypothesis considered "proved". As a result, various mathematicians have claimed to have proven the Hypothesis, and others have claimed to have disproven it, but so far all of these claims have been false.

One of the earliest claims of proving the Riemann Hypothesis was that of the Dutch-born French mathematician Thomas Stieltjes (pronounced STEEL-ches). In 1885, he said that he had proven a ***stronger*** conjecture than the Riemann Hypothesis—a conjecture which, if true, would lead to the Hypothesis being true. However, his proof never surfaced, and the same conjecture has since been proven false (Goodman and Weisstein). Undoubtedly Stieltjes found a mistake in his reasoning after making his announcement.

More recently, Louis de Branges, a professor at Purdue University, spent many years pursuing a proof of the Riemann Hypothesis by working on a more generalized version of the problem. In 2003, he claimed to have a proof, but did not publish it—the approach that he had taken was flawed, as others had demonstrated as early as 1998 (Goodman and Weisstein).

The opposite situation occurred in 1943: a German mathematician (Hans Rademacher, then a professor at the University of Pennsylvania) nearly published a *disproof* of the Hypothesis. He withdrew his claim at the last moment, when colleagues checking his work discovered a mistake in his paper (Sabbagh, 108-109).

Of course, false proofs are not only written by famous mathematicians. Amateurs often send articles to mathematics journals, claiming to prove—or sometimes disprove—the Riemann Hypothesis. One journal received two papers from the same author within a week. One was a proof of the Hypothesis; the other was a disproof. The author also included a letter to the journal, asking which of the two papers was correct (Sabbagh, 112).

Despite all these failures, it is true that progress has been made in the quest to prove the Riemann Hypothesis. In 1914, G. H. Hardy, a professor at Cambridge and later, Oxford, proved that the zeta function has an infinite number of zeros with real part $\frac{1}{2}$. This result was an important step towards proving the Riemann Hypothesis, but does not prove that *all* non-trivial zeros have real part $\frac{1}{2}$ —we might also have infinitely many zeros that do *not* have real part $\frac{1}{2}$. Also, around 1990, several mathematicians found that at least 40% of the non-trivial zeros will lie on the critical line. Again, this was an important result, despite not proving the Hypothesis (Goodman and Weisstein).

The mathematical community's passion for the Riemann Hypothesis is sometimes played on. One famous example occurred in 1997 when Enrico Bombieri, a professor at the Institute for Advanced Study in Princeton and winner of a Fields medal (the math equivalent of a Nobel Prize), sent his friends an email. A young physicist had seen how to use "the physics corresponding to a near-absolute zero ensemble of a mixture of anyons [a type of elementary particle] and morons [sic] with opposite spins" to solve the Riemann Hypothesis! Word spread around the globe within a few days, and mathematicians around the world eagerly awaited details of the proof (du Sautoy, 2-4). Unfortunately, the email was revealed to be an over-successful April Fool's Day joke. Yet although the joke revolved around physics solving the Hypothesis, this is not as absurd an idea as it may sound. Fields other than number theory may just be the key to proving the Riemann Hypothesis.

Understanding the Riemann Zeta Function

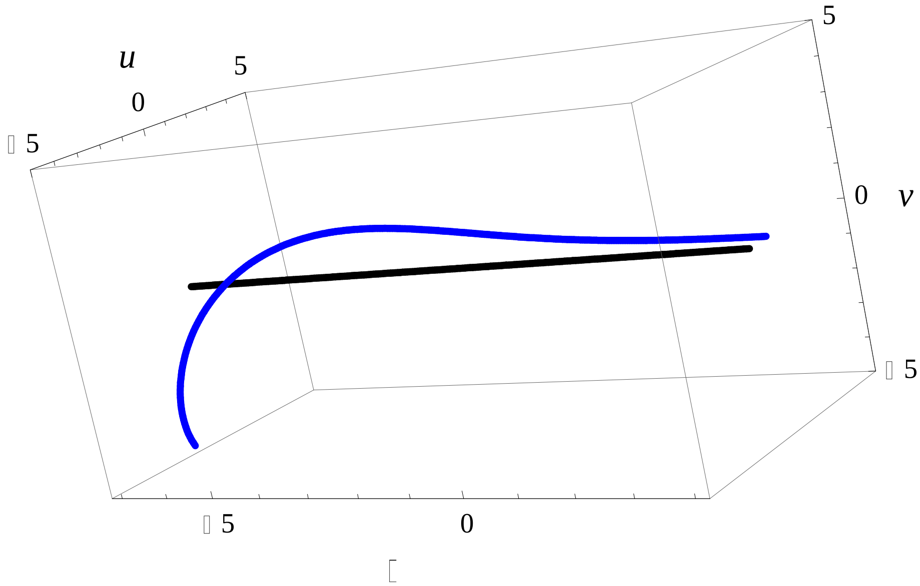
The very definition of the Riemann Hypothesis revolves around Riemann's zeta function. As a result, studying the zeta function's behavior will be important to any proof of the Hypothesis.

Graphs

We stated earlier that any complex number can be graphed in the two-dimensional complex plane. In a similar way, we can graph the Riemann zeta function $\zeta(s)$ in four dimensions—two for the real and imaginary parts of s and another two for the real and imaginary parts of $\zeta(s)$. But here we run into a problem. We cannot physically draw more than three dimensions…so how can we create a picture of the four-dimensional structure of $\zeta(s)$?

We do so by effectively reducing the number of dimensions to three—by holding one of the real or imaginary parts constant. In this way we observe “snapshots” of $\zeta(s)$ –if σ is the real part of s and t is its imaginary part, then for certain values of t, we can observe how the value of $\zeta(s)$ changes as σ varies and t stays at its predetermined value.

In Figure 5, the u- and v-axes are the real and imaginary axes, respectively, of the plane that $\zeta(s)$ is plotted in. We then observe how the value of $\zeta(s)$ changes as σ changes. The black line shows where $u = v = 0$ (that is, where we would have $\zeta(s) = 0$). The blue curve representing the zeta function curls around but never meets that line, so for $t = 6.6$ and values of σ shown (from -7 to 4), the zeta function $\zeta(s) = \zeta(\sigma + it)$ has no zeros.

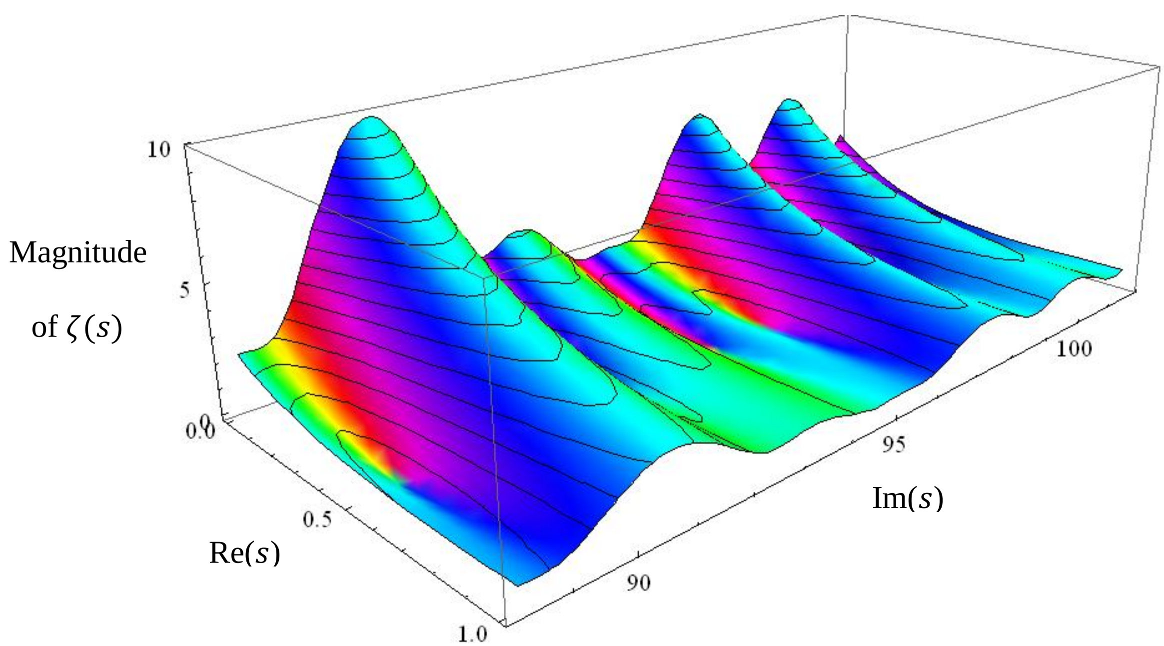


Figure[8] 5: a three-dimensional graph of the zeta function when the imaginary part of s is $t = 6.6$.

We can also make two-dimensional graphs of the zeta function. For example, we can hold both the real and imaginary parts of s constant, making s a constant. This allows us to graph the unique value of $\zeta(s)$ in the complex plane. But because we only see a single value of

$\zeta(s)$, such a graph does not give us much information about the overall structure of the zeta function. A more useful way to view the zeta function in two dimensions is to plot many values of $\zeta(s)$ in the complex plane and observe how they change as s varies[9]—this does not require us to look at "snapshots" of the zeta function's graph.

A third way to graph the zeta function also does not require holding a real or imaginary part of s (or of $\zeta(s)$) constant. Instead of keeping one part constant, we represent it using color, as in the graph below. (Figure 7 actually does not show the real and imaginary parts of $\zeta(s)$. Instead, it shows two other numbers that uniquely define the complex number $\zeta(s)$—its ***magnitude*** and ***argument***. In the complex plane, the magnitude of a complex number z is the distance between the origin and the point representing z. The argument of z is the angle measured counterclockwise from the positive real axis to the aforementioned point.)



Figure[10] 7: A graph of the zeta function.

The different colors indicate the value of the argument of $\zeta(s)$.

By analyzing graphs of all three types, we can gain an understanding of what the true, four-dimensional structures of the Riemann zeta function looks like.

Related Functions

There are many functions similar to the Riemann zeta function for which we have "Riemann Hypotheses." For example, in 1940, the French mathematician André Weil (pronounced "Vay") proved the "Riemann Hypothesis for curves over finite fields." His proof showed that for a certain family of functions graphically similar to the zeta function, a function of that family must have its zeros lie on a straight line (du Sautoy, 295).

Other than functions that *look* similar to Riemann's zeta function, there are also functions that are *defined* similarly[11]. For any one of these functions, if we could prove that it only has non-trivial zeros on the critical line, we would say that it satisfied the Riemann Hypothesis. Unfortunately, we have not proved this result for any of these similarly-defined functions. However, their study may lead to the discovery of new techniques—just as Weil's work did (Thomas)—which may be of use in proving the original form of the Riemann Hypothesis (Conrey, 347-348).

Thus, it may be that a function that is graphically similar to the zeta function will help us to prove the Riemann Hypothesis. But it is equally possible that the key part of a proof will come from the relationship between the zeta function's zeros and the fields of physics and probability.

Another Tutorial

This section will provide some background information on topics related to physics: energy levels, the theory of quantum chaos, and matrices.

Energy Levels

Energy is ***quantized***. This means that any amount of energy must be composed of a whole number of tiny packets—quantums—of energy. Thus, as the energy of an object increases, it does not increase continuously or smoothly. Rather, it increases with tiny, jerky additions of one quantum of energy at a time. (This is analogous to walking up a staircase, step by step, instead of riding up an escalator.) Note that the quantization of energy is only apparent at the atomic level—for macroscopic objects, it can easily seem as if energy increases continuously.

In addition, sometimes an object cannot increase its energy by exactly one quantum. For example, an atom is composed of a nucleus and several electrons orbiting the nucleus. The energy which the atom stores within itself comes from the interaction of each particle with the other particles around it; because of this interaction, the atom's energy can only take on certain values, which may be separated by more than one quantum. (To extend the staircase analogy, imagine that several staircases are "added" to each other to create a staircase with uneven steps—the distances between consecutive steps may be several times that between the stairs on the original staircase. These steps represent the *energy levels* of the atom—the values that its energy can take on.)

But it is not only atoms that have energy levels. The energy of *any* system of two or more particles can only take on certain values, because the system's individual particles interact in such a way to make other energy values impossible to reach. For example, atomic nuclei are composed of protons and neutrons, so such nuclei have energy levels. In addition, when an electron is in a space bounded by other particles, the system composed of those particles and the electron has energy levels.

Quantum Chaos

In physics, a certain system is ***chaotic*** if a slight initial change causes the system to be very different. For example, weather is chaotic—air moving in one direction might cause a gust of wind, while a slight change in its direction could cause a tornado. The word "quantum" refers to quantum mechanics (which studies the behavior of atoms and subatomic particles)—so quantum chaos refers to the combination of the concepts of chaos and quantum mechanics (Gutzwiller).

Matrices

A ***matrix*** is an array of numbers, such as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A ***random matrix*** is a matrix whose elements are picked randomly from some previously-chosen range of numbers (which varies for different types of random matrices).

If a matrix is ***square*** (the number of columns is equal to the number of rows), then it has a ***characteristic polynomial***. The zeros of the characteristic polynomial are the matrix's ***eigenvalues***[12].

Matrices are related to physics because the eigenvalues of certain random matrices behave quite similarly to experimentally-observed energy levels—when two energy levels are far apart, the corresponding eigenvalues tend to be far apart, and when the energy levels are close, the corresponding eigenvalues are usually close to each other.

The Zeta Function's Zeros and Physics

Recall that an equivalent form of the Riemann Hypothesis is that if s is rewritten as $\frac{1}{2} + it$, non-trivial zeros of $\zeta(s)$ only occur when t is real. (This is the form in which Riemann originally proposed the Hypothesis.) In other words, if we define a new function ξ such that $\xi(t) = \zeta\left(\frac{1}{2} + it\right)$, all the non-trivial zeros of $\xi(t)$ should be real. (ξ is the Greek letter xi and can be pronounced either "zai" or "ksai.") Thus, if we can relate $\xi(t)$ and its zeros to a function that only has real zeros, it may be possible to prove that the zeros of $\xi(t)$ itself are real, and thereby prove the Riemann Hypothesis. This approach is a popular one; in particular, many mathematicians have tried relating the zeros of $\xi(t)$ to the eigenvalues of certain matrices.

In the early 1900s, two mathematicians independently suggested that the non-trivial zeros of $\xi(t)$ could be the eigenvalues of a special type of matrix: a ***Hermitian*** matrix[13] (Derbyshire, 277). (This idea is now called the Hilbert-Pólya Conjecture, after the two mathematicians who first proposed it.) If this conjecture is true, then the Riemann Hypothesis will immediately follow, because Hermitian matrices have the property that all their eigenvalues are real!

Then, in 1972, the American number theorist Hugh Montgomery made a discovery that not only supported the Hilbert-Pólya Conjecture but also linked the Riemann zeta function and Hypothesis to physics. Montgomery had been studying the zeros of the zeta function—specifically, the distances between those consecutive, non-trivial zeros that lie on the critical line. (Note that these distances correspond exactly to those between the real values of t such that

$\xi(t) = 0$.) Montgomery eventually discovered an expression that represented the statistical distribution of those distances, and after a chance meeting, told his result to physicist Freeman Dyson, of the Princeton Institute for Advanced Study. Dyson immediately said that Montgomery's expression was the same as one used to study the behavior of the differences between eigenvalues of certain random Hermitian matrices (Thomas)—a subject that Dyson was familiar with because such eigenvalues are used to represent the energy levels of heavy atomic nuclei!

Based on this revelation, Montgomery hypothesized that *all* the statistics of real zeros of $\xi(t)$ will match the corresponding statistics of eigenvalues of random Hermitian matrices. If this is true, then the xi function's zeros—and by extension, the imaginary parts of some of the zeta function's zeros—probably represent the energy levels of some physical object (Conrey, 349).

Of course, we may wonder what type of material this physical object may be. One possible answer comes from the zeta function's link to quantum chaos. During the 1970s, physicists began studying the energy levels of an electron (which are described by quantum mechanics) while the electron is enclosed in a small space. Depending on the space's shape, the electron's path may be regular or chaotic. If the path is regular, no relationship to the zeta function is seen, but if it is chaotic, the spacing between the electron's energy levels behaves *exactly* like the spacing between the xi function's zeros (du Sautoy, 276-279). In fact, the similarity between the two is even stronger than that between the zeros and the energy levels of heavy nuclei! This makes it even more likely that the zeta function's zeros (each of which corresponds uniquely to one of the xi function's zeros) are related to some type of physical object…one that can be described using quantum chaos. No one yet knows what this object could be, although Freeman Dyson suggests a type of quasi-crystal (a sort of matter that has neither a perfectly ordered nor completely disordered structure).

Thus, there is a fascinating link between the zeta function and energy levels. But this is not interesting merely because of the resulting connections between physics, the zeta function, number theory, and prime numbers—if the xi function's zeros are the energy levels of an actual object, then they must be real…and so the Riemann Hypothesis will be proved. In this way, physical objects may very well be the key to proving the Riemann Hypothesis.

The Riemann Hypothesis and the Randomness of Primes

As we saw in the previous section, the behavior of the Riemann zeta function is related to that of random matrices. Can it be that the Riemann Hypothesis is related to randomness?

In fact, it is—through its relationship to prime numbers. The Hungarian mathematician Paul Erdős and the Polish statistician Mark Kac discovered this result: suppose that N is the number of primes dividing a positive integer x. The chance that x's value of N will fall in a certain range can be approximated by a certain formula—one saying that those x with a certain value of N are randomly sprinkled among all the positive integers (Guiasu, 111). In other words, the primes, which all possess $N = 1$, have aspects of random behavior.

A word of caution here: the distribution of prime numbers among the positive integers is *not* completely random. For example, between any number and its double, there must be at least one prime number. (This result is known as Bertrand's Postulate or Chebyshev's Theorem.) The property that Erdős and Kac discovered means that when we consider the infinitely large interval of positive integers, the primes are randomly distributed within that interval. On intervals of finite size, no matter how large they are, we cannot be sure whether the behavior of prime numbers is random, or whether it actually follows a pattern.

As an analogy, consider a simplified weather system. Assume that it can only be sunny or rainy on any given day, and compare the number of sunny and rainy days within a certain period of time. We know that in the long run any day is equally likely to be sunny or rainy—rainy days are randomly dispersed among all of time. We also know that the distribution of rainy days is not *truly* random—suppose that it is known that we must have *some* rain every month. But given these restrictions, are rainy days randomly scattered within a single year? Or does it rain on many days, but always, consistently, on every 10^{th} day?

This is the issue with the distribution of the prime numbers…on small intervals it appears to be random and patternless, but there could be a hidden rule governing it. So do prime numbers form a pattern? Or do they behave within the limits of randomness—are they as random as they can be, given the restrictions (such as Bertrand's Postulate) that they are subject to?

The answers to these questions are promised by the Riemann Hypothesis. If the Hypothesis is true and all of the zeta function's non-trivial zeros lie on the critical line, the prime numbers are distributed in a mostly random way. But if zeros occur off the critical line, then the prime numbers are ***biased***—they will form a pattern rather than a random collection of numbers. The farther the zeros are from the critical line, the more biased—and orderly—the prime numbers' distribution is (du Sautoy, 163-167).

Moreover, this idea means that if the prime numbers are arranged randomly, the Riemann Hypothesis is true. Of course, we cannot simply look at the prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,… and say that there is no pattern. But mathematicians who work with prime numbers *have* concluded that there is no obvious pattern to the prime numbers, a fact supported by the knowledge that almost all of the zeta function's zeros lie very close to the critical line (Conrey, 344). Thus, although we do not know if there is a pattern to the prime numbers, we have the consolation of knowing that if a pattern exists, it is probably very slight—and if a zero of the zeta function lies off of the critical line, it is probably still very close.

Why Address the Riemann Hypothesis?

In the almost 155 years since the Riemann Hypothesis was first proposed, countless formulas have been created that begin with the words "if the Riemann Hypothesis is true." Not the least of these is a result proved in 1901 by the Swedish mathematician Helge von Koch (who also described the Koch snowflake fractal):

If the Riemann Hypothesis is true, $\pi(x) = Li(x) + O(\sqrt{x}\ln x)$

(The prime counting function, written as $\pi(x)$, gives us the number of primes less than or equal to x. Note that this function is unrelated to the number $\pi \approx 3.14$.)

The above expression actually gives us the error bound in the Prime Number Theorem: it turns out that $Li(x)$ (the logarithmic integral[14]) is the expression that $\pi(x)$ approaches, and so the amount by which $\pi(x)$ can differ from $Li(x)$ is $O(\sqrt{x}\ln x)$.

Thus, von Koch's result seems to give an explicit formula for the prime counting function $\pi(x)$. Unfortunately, it does not. Although Li is a function, big-Oh, (the symbol in the term $O(\sqrt{x}\ln x)$) is not—it does not output values. Instead, the expression

$$\pi(x) = Li(x) + O(\sqrt{x}\ln x)$$

means[15] that for large enough x,

$$\pi(x) \leq Li(x) + c\sqrt{x}\ln x$$

for some positive (and real) constant c. From this expression we do not know the exact value of $\pi(x)$, but rather the way the function behaves. For large enough x, $\pi(x)$ will never grow faster than $Li(x)$ plus some multiple of $\sqrt{x}\ln x$.

This seems a trivial enough result, but it is the tightest known bound on $\pi(x)$—and it depends on the validity of the Riemann Hypothesis. Thus, if the Riemann Hypothesis is true, we are one step closer to figuring out exactly how $\pi(x)$ behaves.

But what is the *practical* point of knowing that the Riemann Hypothesis is true? Yes, all those mathematical formulas will be validated, but will humans gain anything by proving the Hypothesis?

In an episode of the American television show NUMB3RS, a character claims that solving the Riemann Hypothesis would destroy internet security (Goodman and Weisstein). But that is fiction. True, prime numbers are essential to cryptography—the security of the RSA method, one of the most commonly-used encryption methods, depends on the difficulty of factoring a number into two extremely large primes. Yet for practical purposes, we already act as if the Riemann Hypothesis is true. Issues would only occur if the Hypothesis were revealed to be false.

The general consensus is that the Riemann Hypothesis is true. As was pointed out by Erdős, if it were false, the primes would be arranged in some type of pattern—and so far we have seen none. Also, some feel that finding even a single non-trivial zero off of the line would be "an aesthetically distasteful situation" (qtd. in du Sautoy, 215) that contradicts Nature's tendency to be elegant. Moreover, that first zero would be an extremely important mathematical constant, due to the zeta function's relationship to prime numbers. Why haven't we encountered this first zero in other formulas? Other mathematicians consider the trillions of zeros that *do* lie on critical line sufficient evidence to believe the Hypothesis (du Sautoy, 218). So the majority of the mathematical community believes that the Riemann Hypothesis is true (Derbyshire, 356). Why can't we just say that the Riemann Hypothesis *seems* true and move on?

The problem is that if the Riemann Hypothesis has not indisputably been proven true, it might still be false, and the fact that trillions of zeros do lie on the critical line does not prevent this from happening. After all, Gauss had hypothesized that his expression for the prime counting function would always be greater than the actual function, but in 1914 (about 60 years after Gauss died) it was discovered that this is not true. Later on, it was found that the first time that Gauss's expression is *smaller* than the prime counting function occurs somewhere around

$$10^{10^{10^{34}}}$$

a gigantic number which is *not* an important mathematical constant in any other formula. A similar turnabout could happen with the Riemann Hypothesis—yes, all known zeros of the zeta function lie on the critical line, but this does not mean that *all* zeros will do so. Perhaps we will find that the quadrillionth zero lies off of the critical line. (One quadrillion is a thousand trillion, or 1 with 15 0's.)

As Enrico Bombieri puts it, "the failure of the Riemann Hypothesis would create havoc in the distribution of prime numbers" (qtd. in Thomas). So mathematicians believe the Riemann Hypothesis to be true, and yet—what if it isn't? What if it is realized that the prim numbers are arranged in a non-random way—a predictable way? Then what would happen when we used those predictable primes to create encryption keys for our credit cards? It is if the Riemann Hypothesis fails that disorder will result. So mathematicians search for whether that disorder will happen. They seek confirmation that the primes are organized in a certain way—a random, chaotic, patternless…yet unique and beautiful way. They seek confirmation of the Riemann Hypothesis.

The Riemann Hypothesis:

All non-trivial zeros of $\zeta(s)$ are of the form $s = \frac{1}{2} + it$.

Notes

1. There are many justifications for classifying 1 as neither prime nor composite. First and foremost, 1 has exactly one divisor, so if we define primes to have two divisors, 1 cannot be prime. In addition, every composite number is the product of two or more primes, so any composite number should have three or more divisors—but 1 only has one, so we should not call it composite either.

For more reasons as to why 1 is not prime, see the article “Why is the number one not prime?” at http://primes.utm.edu/notes/faq/one.html.

2. For more formulas defining the zeta function, see http://functions.wolfram.com/ZetaFunctionsandPolylogarithms/Zeta/

3. Why do we use σ (sigma) and t in our representation of complex numbers? We could use any two variables to represent the real and imaginary parts of z—for example, we could write $z = x + iy$. However, we sometimes use x and y as independent numbers (in which case x and y may have nonzero imaginary parts). So by following the convention of writing $z = \sigma + it$, we emphasize that σ and t are *real* numbers.

4. Figure 2 (slightly altered) from http://mathworld.wolfram.com/CriticalLine.html**.**

5. Figure 3 (slightly altered) from http://mathworld.wolfram.com/CriticalStrip.html.

6. Parts of Figure 4 from http://mathworld.wolfram.com/CriticalStrip.html.

7. The Prime Number Theorem is intimately related to the zeta function—in fact, it is equivalent to the statement that the non-trivial zeros of the Riemann zeta function never have real part 1 (Goodman and Weisstein).

8. Figure 5 generated from the interactive Wolfram Demonstration “The Riemann Zeta Function in Four Dimensions” at http://demonstrations.wolfram.com/

TheRiemannZetaFunctionInFourDimensions/

9. For an interactive graph showing how the values of $\zeta(s)$ change as s moves along the critical line, see the Wolfram Demonstration at http://demonstrations.wolfram.com/ValueOfTheZetaFunctionAlongTheCriticalLine/

10. Figure 7 generated from the interactive Wolfram Demonstration "Riemann Zeta Function near the Critical Line" at http://demonstrations.wolfram.com/RiemannZetaFunctionNearTheCriticalLine/

11. The Riemann zeta function is a special instance of a zeta function—all zeta functions are defined similarly to Riemann's zeta function. There are also certain functions called L-functions (because they are defined by L-series) that are special cases of zeta functions and are also similar to the Riemann zeta function. For more information, see the "Zeta Function" article at Wolfram MathWorld: http://mathworld.wolfram.com/ZetaFunction.html.

12. For more information about characteristic polynomials and eigenvalues, see the "Eigenvalue" article at Wolfram MathWorld: http://mathworld.wolfram.com/Eigenvalue.html.

13. A Hermitian matrix is a matrix such that numbers that are "opposite" each other across the matrix's main diagonal (below, dashed line) are ***complex conjugates*** of each other. That is, if one element is $a + bi$, the element opposite it is $a - bi$. An example of a Hermitian matrix is:

The Main Diagonal

$$\begin{bmatrix} \mathbf{2} & 1+i & 3-2i & 1.5i \\ 1-i & \mathbf{5} & 4 & 1.003+5.02 \\ 3+2i & 4 & \mathbf{-1} & -0.423-3i \\ -1.5i & 1.003-5.02i & -0.423+3i & \mathbf{0} \end{bmatrix}$$

Because each entry on the main diagonal (bolded elements) is "opposite" itself, it must be its own complex conjugate, and so it must be real. However, other entries are not required to be real; they can have imaginary parts or even be "pure imaginary" numbers. In addition, a Hermitian matrix must be ***square*** (or else the main diagonal will not exist).

One special property of Hermitian matrices is that their eigenvalues must be real.

14. $Li(x)$ is the "European" definition of the logarithmic integral (see the Wolfram Mathworld "Logarithmic Integral" article at http://mathworld.wolfram.com/LogarithmicIntegral.html):

$$Li(x) = \int_{2}^{x} \frac{1}{\ln u} du$$

(The "American" definition of the logarithmic integral is represented by $li(x)$ and takes the integral from 0 to x.)

15. Strictly speaking,

$$\pi(x) = Li(x) + O(\sqrt{x}\ln x)$$

means that for large enough x and some positive, real constant c,

$$|\pi(x)| \leq Li(x) + c\sqrt{x}\ln x$$

However, the prime counting function $\pi(x)$ is always positive, so we can ignore the absolute value signs.

Appendix A: Selected Proofs

In this Appendix we shall prove certain results relevant to the Riemann zeta function, in particular concerning prime numbers and the behavior of the zeta function's zeros.

Proof 1: There are Infinitely Many Primes

The following proof was discovered by Euclid around 300 B.C.E., and is a classic example of proof by contradiction. First, suppose that there exist only n primes: $2, 3, 5, \ldots, p_n$. (p_n is the nth prime.)

Let $N = 2 \times 3 \times 5 \times \ldots \times p_n + 1$. N is not equal to any of the n primes and is a positive integer greater than 1, so it must be composite. In other words, N must be divisible by at least one of the n primes.

However, for any one of the n primes, N is one more than a multiple of that prime. So N itself cannot be a multiple of that prime—N is not a multiple of *any* of the n primes. Contradiction.

This contradiction implies that our original premise ("suppose that there exist only n primes") is false. Thus, we do not have a finite number, n, of primes: rather, **there are infinitely many primes**.

Proof 2: Euler's 1737 Result

Here we derive Euler's 1737 result, which relates the zeta function (here only dealing with real arguments) to the ***Euler product*** (the infinite product shown below):

If s is a real number greater than 1,

$$\boldsymbol{\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}}$$

Our proof is as follows: first, note that

$1 + 2 + 3 + 4 + 5 + 6 + \cdots$

$$= (1 + 2^1 + 2^2 + \cdots)(1 + 3^1 + 3^2 + \cdots)(1 + 5^1 + 5^2 + \cdots) \ldots$$

where the sum on the left is the sum of all positive integers, and the product on the right is the product, taken over all primes p, of the infinite geometric series $1 + p^1 + p^2 + p^3 + p^4 + \cdots$. (This is the case because every positive integer has a unique ***prime factorization***: a representation as the product of powers of prime numbers.)

In a similar way,

$$\sum_n \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

$$= \left(\frac{1}{1} + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} \ldots\right)\left(\frac{1}{1} + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} \ldots\right)\left(\frac{1}{1} + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} \ldots\right)$$

$$= \prod_P \left(\frac{1}{1} + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

Rewriting the infinite geometric series within the parentheses, we have

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

provided that

$$\left|\frac{1}{p^s}\right| < 1$$

Since p is positive, p^s is positive, so we can rewrite this inequality as

$$p^s = |p^s| > 1 \quad \longrightarrow \quad 0 < s$$

Thus, if $s > 0$, our rewriting of the infinite product is justified, and

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

is true algebraically. Numerically, however, we cannot say that the two expressions are equal unless they are finite numbers—specifically, unless the infinite sum converges.

From a well-known result in calculus, the sum

$$\sum_n \frac{1}{n^s}$$

converges if and only if $s > 1$.

Thus, the equation

$$\boldsymbol{\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}}$$

holds when the real number s is greater than 1.

Proof 3: Extension of Euler's 1737 Result

We now generalize Euler's 1737 result to situations where s may be nonreal:

$$\boldsymbol{\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}}$$

if the complex number s has real part greater than 1.

This proof is very similar to Proof 2; in particular, we have

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

provided that

$$\left|\frac{1}{p^s}\right| < 1$$

and that the infinite sum on the left converges.

Suppose that s has real part σ and imaginary part t, and suppose that x is a positive real number. Since x is positive, x^σ is also positive. In addition, it is known that the absolute value of x^{it} is 1 (although the details are beyond the scope of this text.) Thus

$$|x^s| = |x^{\sigma+it}| = |x^\sigma| \times |x^{it}| = |x^\sigma| \times 1 = x^\sigma$$

Since all primes p are positive real numbers,

$$|p^s| = p^\sigma$$

and so we wish to have

$$\left|\frac{1}{p^s}\right| = \frac{1}{p^\sigma} < 1$$

which, by logic similar to that in Proof 2, only holds if $\sigma > 0$.

Moreover, it is known that if

$$\sum_n \frac{1}{n^\sigma} = \sum_n \left|\frac{1}{n^s}\right|$$

converges, then

$$\sum_n \frac{1}{n^s}$$

will too. (A word of caution: the reverse is not necessarily true.) Since it is known (using calculus) that

$$\sum_n \frac{1}{n^\sigma}$$

converges for $\sigma > 1$, this means that

$$\sum_n \frac{1}{n^s}$$

will converge if $\text{Re}(s) = \sigma > 1$.

Therefore

$$\boldsymbol{\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}}$$

if the real part of s is greater than 1.

Proof 4: $\zeta(s)$ Has No Zeros With Real Part Greater Than 1

From Proof 3, we know that when $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

Irrespective of the value of s, we can rearrange the fraction on the right side as

$$\frac{1}{1 - \frac{1}{p^s}} = \frac{1}{\frac{p^s - 1}{p^s}} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

which we note will only equal 0 if

$$\frac{1}{p^s - 1} = -1$$

that is, if and only if $p^s = 0$, which in turn can only be 0 if $p = 0$. (As an informal proof of this statement, we observe that doing exponentiation with a nonzero base—multiplying the base by itself several times—should not result in 0.)

However, all primes p are positive—nonzero. Thus $p^s \neq 0$ for all p and s. So

$$\frac{1}{1 - \frac{1}{p^s}}$$

can never be 0, and

$$\prod_p \frac{1}{1-\frac{1}{p^s}}$$

is never zero.

Now, when $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

and the product on the right can never be 0. Thus **when $\mathbf{Re}(\boldsymbol{s}) > \mathbf{1}$, $\boldsymbol{\zeta}(\boldsymbol{s}) \neq \mathbf{0}$.** In other words, **the zeta function has no zeros to the right of the critical strip.**

Proof 5: $\zeta(s)$ Has No Non-trivial Zeros With Real Part Less Than 0

Our starting point is the functional equation of the zeta function:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

Since both 2 and π are nonzero, $2^s$and π^{s-1} can never be 0. In addition, because of the definition of the gamma function, $\Gamma(1-s)$ is always nonzero. (See http:// mathworld.wolfram.com/GammaFunction.html for more information.) In addition, from Proof 4 in Appendix A, $\zeta(1-s) \neq 0$ when $\mathrm{Re}(1-s) > 1$, or when $\mathrm{Re}(s) < 0$.

Thus, if $\mathrm{Re}(s) < 0$, all the factors on the right side of

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

except for $\sin\left(\frac{\pi s}{2}\right)$ are nonzero. Thus $\zeta(s)$ can be 0 if and only if $\sin\left(\frac{\pi s}{2}\right) = 0$…but the values of s for which this happens are simply the trivial zeros of the zeta function (see Appendix B for the full derivation of the trivial zeros). This means that **the zeta function has no non-**

trivial zeros with real part less than zero—that is, **it has no non-trivial zeros to the left of the critical strip.**

Appendix B: Derivation of the Trivial Zeros of the Zeta Function

We know that for all (potentially nonreal) $s \neq 1$, the following functional equation holds:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-\mathrm{s}) \zeta(1-s)$$

This means that $\zeta(s) = 0$ when $\sin\left(\frac{\pi s}{2}\right) = 0$, provided that every factor on the right side (2^s, π^{s-1}, and so on) is a finite number. (If a factor is infinite, then $\zeta(s)$ is of the form $0 \times \infty$, which is the product of an infinite number of infinitesimally small quantities…a product which is not always 0.) The behavior of the sine function is such that it equals zero every time its argument (the part inside the parentheses) is a multiple of π, which in this case happens when s is a multiple of 2.

This suggests that (real) multiples of 2 should be considered trivial zeros. Is this the case?

We know from Proof 4 in Appendix A that $\zeta(s)$ has no zeros when the real part of s is greater than 1, so we need not even consider the case where s is a positive multiple of 2 (since this causes the real part of s—s itself—to be greater than 1).

In addition, when $s = 0$, $\zeta(1-s) = \zeta(1)$ is infinite, leading $\zeta(0)$ to be of form $0 \times \infty$. This means that $s = 0$ may not be a zero of the zeta function (and in fact, it turns out not to be).

So what about negative multiples of 2? When s is a negative integer, $1 - s$ is a positive integer, so both $\Gamma(1-s)$ and $\zeta(1-s)$ are finite quantities. (This is since if n is a positive integer, $\Gamma(n) = (n-1)!$, which is finite. In addition, $\zeta(1-s)$ is finite provided that $s \neq 0$.)

Moreover, for $s < 0$, we know that 2^s, π^{s-1}, and $\sin\left(\frac{\pi s}{2}\right)$ are all finite numbers.

Thus when s is a *negative multiple of 2*, $\sin\left(\frac{\pi s}{2}\right) = 0$ and all factors on the right side of

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

are finite, which means that $\zeta(s) = 0$. So **the trivial zeros of the zeta function are the negative multiples of 2.**

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